Modulo 2 Conway polynomials of rational links

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Abstract. We show that a polynomial is the modulo 2 Conway polynomial of a rational link if and only if it is a Fibonacci polynomial modulo 2. We deduce a simple proof of the Murasugi characterization of the modulo 2 Alexander polynomials of rational knots. We also deduce a fast algorithm to test when the Alexander polynomial of a rational knot K is congruent to 1 modulo 2, which is a necessary condition for K to be Lissajous.

1. Introduction

J. W. Alexander (1928), and independently K. Reidemeister, discovered the first effectively calculable knot invariant, now called the Alexander polynomial. It is still one of the most useful of all knot invariants.

Rational knots (or links) are a very important and simple class of knots. They are the knots which have a representation such that the abcissa has only two local maxima and two local minima (see figure 1). For each rational number $\frac{\alpha}{\beta}$ Schubert has constructed a

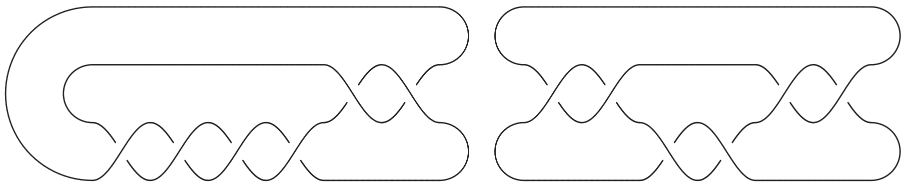

Figure 1. The Stevedore knot 6_1 and the Fibonacci link $\mathcal{F}_2^{(3)}$

rational link denoted $S(\frac{\alpha}{\beta})$. This link is a knot if α is odd, and a two component link if α is even. Spectacularly, he proved that every rational knot (or link) is of this form (see [16, 9, 8]). He also proved that the links $S(\frac{\alpha}{\beta})$ and $S(\frac{\alpha'}{\beta'})$ are isotopic if and only if $\alpha = \alpha'$ and $\beta' \equiv \beta^{\pm 1} \pmod 2$ (see [16, 9, 8]).

J. H. Conway discovered a simple method to compute the Alexander polynomial of a knot (see [3, 7]). By a simple change of variables, he deduced the Alexander polynomial from a simpler one, now named the Conway polynomial. His method provides an easy algorithm (theorem 8) to deduce the Conway polynomial of a rational knot (or link) from its Schubert fraction $\frac{\alpha}{\beta}$.

In this note, we shall study the converse problem: given a polynomial, is it the Conway polynomial of a rational knot?

We obtain the following result (where the Fibonacci polynomials are defined as in [19] by:

$$f_0 = 0, f_1 = 1, f_{n+2}(z) = z f_{n+1}(z) + f_n(z), \tag{1}$$

Theorem 1. *Let $P(z) \in Z[z]$ be the Conway polynomial of a rational knot (or link). There exists a Fibonacci polynomial $f(z)$ such that $P(z) \equiv f(z) \pmod 2$.*

Then, we deduce a simple proof of a beautiful criterion due to Murasugi ([14, 2])

Corollary 2 (Murasugi (1971))**.** *Let $P(t) = a_0 - a_1(t+t^{-1}) + a_2(t^2+t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})$ be the Alexander polynomial of a rational knot. There exists an integer $k \le n$ such that $a_0, a_1, \ldots, a_k$ are odd, and $a_{k+1}, \ldots, a_n$ are even.*

Since the Conway polynomial of the torus link $\mathrm{T}(2, n)$ is the Fibonacci polynomial $f_n(z)$ (see [11]), we see that these results are in fact characterizations of the modulo 2 Conway and Alexander polynomials of rational knots.

The simplicity of our computations provide an easy algorithm for the modulo 2 Conway (and Alexander) polynomials of rational knots.

Theorem 3. *Let K be a rational link (or knot) of Schubert fraction $\dfrac{\alpha}{\beta} = [2b_1, 2b_2, \ldots, 2b_m]$. Let us define the sequence e_i by $e_0 = 0$, $e_i = (e_{i-1} + 1 + b_i) \bmod 2$, $i = 1, \ldots, m$.*

Then the modulo 2 Conway polynomial of K is the Fibonacci polynomial $f_D(z)$, where

$$D = \left| \sum_{i=1}^{m} (-1)^{e_i} \times (b_i \bmod 2) - e_m \right|.$$

This algorithm may be used to easily see that some rational knots cannot be Lissajous knots.

2. The Conway and Alexander polynomials

The Conway polynomial of a knot (or link) K is denoted by $\nabla_K(z) = c_0 + c_1 z + \cdots + c_n z^n$. The Conway polynomials of knots are characterized by the fact that they are even polynomials such that $\nabla_K(0) = 1$. The Alexander polynomial $\Delta_K(t)$ is deduced from the Conway polynomial:

$$\Delta_K(t) = \nabla_K\Big(t^{1/2} - t^{-1/2}\Big).$$

The Alexander polynomial of a knot is a Laurent polynomial. The following result allows us to recover the Conway polynomial of a knot from its Alexander polynomial.

Lemma 4. *If $z = t^{1/2} - t^{-1/2}$, and $n \in \mathbf{Z}$ is an integer, we have the identity*

$$f_{n+1}(z) + f_{n-1}(z) = (t^{1/2})^n + (-t^{-1/2})^n,$$

where $f_k(z)$ are the Fibonacci polynomials.

Proof. Let $A = \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix}$ be the (polynomial) Fibonacci matrix. If $z = t^{1/2} - t^{-1/2}$, the eigenvalues of A are $t^{1/2}$ and $-t^{-1/2}$, and consequently $\operatorname{Tr} A^n = (t^{1/2})^n + (-t^{-1/2})^n$. On the other hand, we have $A^n = \begin{bmatrix} f_{n+1}(z) & f_n(z) \\ f_n(z) & f_{n-1}(z) \end{bmatrix}$, and then $\operatorname{Tr} A^n = f_{n+1}(z) + f_{n-1}(z)$. ☐

From this lemma, we immediately deduce:

Corollary 5. *Let the Laurent polynomial $P(t)$ be defined by*

$$P(t) = a_0 - a_1(t+t^{-1}) + a_2(t^2+t^{-2}) - \cdots + (-1)^n a_n(t^n+t^{-n}).$$

We have

$$P(t) = \sum_{k=0}^{n} (-1)^k (a_k - a_{k+1}) f_{2k+1}(z),$$

where $z = t^{1/2} - t^{-1/2}$, and $a_{n+1} = 0$.

The following formula is an immediate consequence of this corollary:

$$f_{2k+1}\big(t^{1/2} - t^{-1/2}\big) = (t^k + t^{-k}) - (t^{k-1} + t^{1-k}) + \cdots + (-1)^k. \tag{2}$$

Example 6. *Let K be a knot of Alexander polynomial $\Delta_K(t) = 1 - (t^2+t^{-2}) + (t^3+t^{-3})$, (for example the 8_{19} knot). Using the lemma, we get its Conway polynomial*

$$\nabla_K(z) = 1 - \big(f_5(z) + f_3(z)\big) + \big(f_7(z) + f_5(z)\big) = 1 - f_3(z) + f_7(z).$$

We shall use the classical notation for continued fractions:

$$[q_1, \ldots, q_m] = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{\ddots + \cfrac{1}{q_m}}}}. \tag{3}$$

Euclidean algorithm with even quotients provides the following continued fractions:

Proposition 7. ([4, p. 207]) *Any fraction $\dfrac{\alpha}{\beta}$, $\alpha \not\equiv \beta \,(\mathrm{mod}\, 2)$, has a continued fraction expansion with even quotients:*

$$\frac{\alpha}{\beta} = [2b_1, 2b_2, \ldots, 2b_m], \quad \textit{where } b_i \in \mathbf{Z}.$$

The only knot theoretic result that we need is the following remarkable theorem.

Theorem 8. [4] *Let $K = S(\dfrac{\alpha}{\beta})$ be a rational knot (or link), and let $\dfrac{\alpha}{\beta} = [2b_1, 2b_2, \ldots, 2b_m]$. The Conway polynomial of K is*

$$\nabla_K(z) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -b_1 z & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_2 z & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} (-1)^m b_m z & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{4}$$

Since it is known that $S(\dfrac{\alpha}{\beta}) = S(\dfrac{\alpha}{\beta - \alpha})$, we can use this theorem to compute the Conway polynomial of any rational knot (or link).

Let us consider some examples.

Example 9 (The torus links). *The torus link $\mathrm{T}(2, m)$ is the rational link (or knot) of Schubert fraction $m = \dfrac{m}{1}$. It is proved in* [7] *and* [11] *that the Conway polynomial of $\mathrm{T}(2, m)$ is $f_m(z)$. If $m = 2k+1$ (*i.e. *$\mathrm{T}(2, m)$ is a knot) we obtain the Alexander polynomial*

$$\Delta(t) = f_{2k+1}\big(t^{1/2} - t^{-1/2}\big) = (t^k + t^{-k}) - (t^{k-1} + t^{1-k}) + \cdots + (-1)^k. \tag{5}$$

Example 10 (The twist knots)**.** *The Twist knot $\mathcal{T}_n$ is the rational knot of Schubert fraction $n+\dfrac{1}{2}$. We have the continued fractions $\dfrac{\alpha}{\beta}=[n,2]=[n+1,-2]$. We immediately get its Conway polynomial $\nabla(z)=1-(-1)^n\lfloor\frac{n+1}{2}\rfloor z^2$. It is congruent to a Fibonacci polynomial modulo 2.*

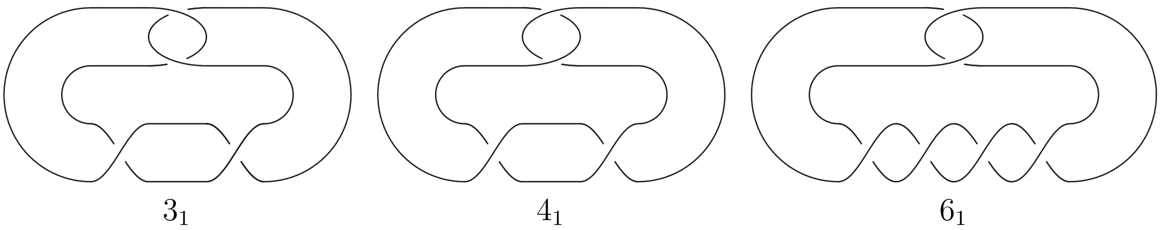

FIGURE 2. Some Twist knots, the trefoil $\overline{\mathcal{T}}_1=3_1$, the figure-eight knot $\mathcal{T}_2=4_1$, and the stevedore knot $\mathcal{T}_4=6_1$.

The Fibonacci knots, introduced by J. C. Turner ([18]), are interesting examples of rational knots. Their Alexander and Conway polynomials are studied in [11] (see also [10]).

3. Proofs

Proof of theorem 1. Let $A=\begin{bmatrix} z & 1\\ 1 & 0\end{bmatrix}$, $J=\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}$. We have $AJ=\begin{bmatrix} 1 & z\\ 0 & 1\end{bmatrix}$, hence $(AJ)^2=\begin{bmatrix} 1 & 2z\\ 0 & 1\end{bmatrix}\equiv\mathbf{Id}\,(\mathrm{mod}\,2)$. We also have $J^2=\mathbf{Id}$, and then $JAJ\equiv A^{-1}\,(\mathrm{mod}\,2)$. We thus see that the modulo 2 matrices A and J generate an infinite dihedral group G.

Consider the matrix $M(b)=\begin{bmatrix} bz & 1\\ 1 & 0\end{bmatrix}$, we have $M(b)\equiv A\,(\mathrm{mod}\,2)$ if b is odd, and $M(b)\equiv J\,(\mathrm{mod}\,2)$ if b is even. Consequently, the matrix product of theorem 8 is equivalent modulo 2 to an element of the dihedral group G, that is to A^m or A^mJ.

Since $A^m=\begin{bmatrix} f_{m+1} & *\\ * & *\end{bmatrix}$ and $A^mJ=\begin{bmatrix} f_m & *\\ * & *\end{bmatrix}$, we conclude that $\nabla_K\equiv f_D\,(\mathrm{mod}\,2)$, recall that the Fibonacci polynomials verify $f_{-n}(z)=(-1)^{n+1}f_n(z)$. ☐

Proof of corollary 2. If K is a rational knot, its Conway polynomial is an even Fibonacci polynomial modulo 2, i.e. of the form $f_{2k+1}(z)$. Since $f_{2k+1}\big(t^{1/2}-t^{-1/2}\big)=(t^k+t^{-k})-(t^{k-1}+t^{1-k})+\cdots+(-1)^k$. The result follows from corollary 5. ☐

Proof of theorem 3. The simplicity of the computations in a dihedral group immediately allows us to obtain the Conway and Alexander polynomials. It clearly suffices to obtain their degrees. Let $\dfrac{\alpha}{\beta}=[2b_1,\dots,2b_m]$. Let us define the sequences e_i and d_i by:

$$e_0=0,\ e_i=(e_{i-1}+1+b_i)\,\mathrm{mod}\,2,\quad d_0=0,\ d_i=d_{i-1}+(-1)^{e_i}\times(b_i\,\mathrm{mod}\,2).$$

Using $JA\equiv A^{-1}J\,(\mathrm{mod}\,2)$, we easily show by induction that

$$P=M(b_1)\cdots M(b_i)\equiv A^{d_i}J^{e_i}\,(\mathrm{mod}\,2).$$

As in the proof of theorem 1, we deduce that $\Delta_K\equiv f_D\,(\mathrm{mod}\,2)$ where $D=|d_m+1-e_m|$. ☐

Example. Consider $\dfrac{\alpha}{\beta} = \dfrac{1828139}{1042750}$ and the rational knot $K = S(\dfrac{\alpha}{\beta})$. One can write

$$\frac{\alpha}{\beta} = [b_1, \ldots, b_{10}] = [2, -4, -20, 2, -2, -12, -2, 4, -12, -4]$$

Using a formula of Stoimenow, we see that the crossing number of K is 59 (see [17]). Using theorem 8 and a computer, we obtain the Conway polynomial of K by evaluating

$$P = M(b_1) \cdots M(b_{10}).$$

The Conway polynomial of K is:

$$\nabla_K(z) = 2880\, z^{10} + 4944\, z^8 + 2304\, z^6 + 158\, z^4 - 61\, z^2 + 1.$$

∇_K may be expressed in terms of Fibonacci polynomials and we obtain:

$$\begin{aligned} \nabla_K(z) &= 2880\, f_{11} - 20976\, f_9 + 68496\, f_7 - 128482\, f_5 + 140969\, f_3 - 62886\, f_1 \\ &\equiv f_3 \pmod 2. \end{aligned}$$

On the other hand we have the simpler computations:

$$P \equiv AJ^2A^2JAJ^3 \equiv A^3JAJ \pmod 2 \equiv A^2 \pmod 2.$$

Our algorithm is a formalization of these last computations. It gives

i	0	1	2	3	4	5	6	7	8	9	10
b_i		1	-2	-10	1	-1	-6	-1	2	-6	-2
$b_i \pmod 2$		1	0	0	1	1	0	1	0	0	0
e_i	0	0	1	0	0	0	1	1	0	1	0
d_i	0	1	1	1	2	3	3	2	2	2	2

We see that the Conway polynomial of our knot is not congruent to 1 modulo 2. Hence, by a theorem of V. F. R. Jones, J. Przytycki and C. Lamm ([6], [12]) it cannot be a Lissajous knot.

We easily obtain the number of knots with Conway polynomial congruent to 1 modulo 2 (compare [1]).

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
rational	1	1	2	3	7	12	24	45	91	176	352	693	1387	2752	5504	10965	21931	43776	87552	174933
$\nabla(t) \equiv 1$	0	0	1	1	2	4	8	13	26	51	97	185	365	705	1369	2675	5233	10211	20011	39221

Table 1. The number of rational knots, rational knots with Conway polynomial congruent to 1 modulo 2.

4. Conclusion

We have found an elementary proof of the Murasugi criterion for rational knots. Furthermore our characterization with Conway polynomials is also valid for links.

There are other classical results on the Alexander polynomials of rational knots. K. Murasugi (1958) showed that the signs of coefficients of Alexander polynomials of alternating knots are alternating, which means that all the a_i are non-negative ([13]). In 1979, R. Hartley showed that the coefficient of the Alexander polynomial of rational knots satisfy the condition $a_0 = \ldots = a_k > a_{k+1} > \ldots > \ldots > a_n > 0$ ([5]). Y. Nakanishi and M. Suketa (1993) obtained upper and lower bounds for $|a_i|$ in terms of $|a_n|$ ([15], see also [2]).

All these results can easily be translated to similar properties of Conway polynomials.

As an application, let us mention the recent study of Lissajous knots. V. F. R. Jones, J. Przytycki (1998)and C. Lamm (1997) showed that the Alexander polynomial of a rational Lissajous knot must be congruent to 1 modulo 2 ([6, 12]). This property is the main tool used by A. Boocher, J. Daigle, J. Hoste and W. Zheng (2009) to prove that some rational knots cannot be Lissajous ([1]). Our algorithm provides a faster method to compute the modulo 2 Alexander polynomials.

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